Intuitive interpretation of Mueller matrices of transmission

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Abstract

Polarization metrology has grown to embrace ever more complicated measurement parameters. From amplitude and phase difference to NSC data acquisition, ellipsometry and polarimetry now commonly measure the complete Mueller matrix. This adaption allows for the simultaneous measurement of a large number of optical parameters at the expense of complex interpretation. Herein, we attempt to break down the Mueller matrix into sections and offer intuitive viewpoint of what each element of the four by four Mueller matrix means.

The Mueller matrix

The Mueller matrix is a real valued four by four matrix that expresses the change in polarization of light from one state to another. Usually, this change is imparted from reflection or transmission of a sample but Mueller matrices are equally applicable to any change in polarization. The Mueller matrix acts upon the incoming Stokes vector expression of the polarization. The Stokes vector is a real valued four element vector.

$$S_{output} = MS_{input}$$

$$S = \begin{bmatrix} I \\ Q \\ U \\ V \end{bmatrix}$$

In the above [Eq1], the Stokes vector which consists of measurements for the intensity of light, I, the intensity of light polarized at 0°/90°, Q, the intensity of light polarized at 45°/135°, U, and the intensity of left/right circularly polarized light, V. The input Stokes vector is transformed into the output Stokes vector via the Mueller matrix. As the Mueller matrix acts upon a four element vector to yield a four element vector, the Mueller matrix must be a four by four element matrix.

$$\begin{bmatrix} I_{output} \\ Q_{output} \\ U_{output} \\ V_{output} \end{bmatrix} = \begin{bmatrix} M_{00} & M_{01} & M_{02} & M_{03} \\ M_{10} & M_{11} & M_{12} & M_{13} \\ M_{20} & M_{21} & M_{22} & M_{23} \\ M_{30} & M_{31} & M_{32} & M_{33} \end{bmatrix} \begin{bmatrix} I_{input} \\ Q_{input} \\ U_{input} \\ V_{input} \end{bmatrix}$$

As shown in [Eq2], the full Mueller matrix expression allows the user to interpret the meaning of a few Mueller matrix elements by examination. For instance, the M_{00} element scales the input intensity to the output intensity so this element can be interpreted as the simple transmittance. The M_{01} element scales the linear polarization at 0°/90° to the output intensity. Hence, M_{01} can be interpreted as the linear extinction at 0°/90°. Extinction here means the ratio of input to output light intensity. The terminology for polarization could use some clarification as a number of similar terms exist; transmission/reflectance, absorbance, extinction, etc.

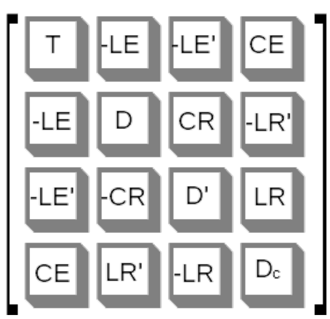
Transmission mean the ratio of light from input to output. Absorption is a measurement of linear attenuation in transmission times the distance of propagation. So here, transmittance is the extrinsic measurement of the ratio of light intensity. While absorption attempts to measure the intrinsic property of linear attenuation of a material in transmission.

Instrinsic property	Extrinsic measurable
Linear birefringence (LB)	Linear retardance (LR)
Circular birefringence (CB)	Circular retardance (CR)
Linear dichroism (LD)	Linear extinction (LE)
Circular dichroism (CD)	Circular extinction (CE)
Absorption (A)	Transmission (T)

In each ideal case above, the intrinsic property can be related to the extrinsic property by integrating over the distance of propagation through the material.

$$\mathrm{LE} = -ln\left(\int_0^l LD(z)dz\right)$$

When the sample under measurement exhibits only a single optical property from the table above, the Mueller matrix can be interpreted from [FIG1]. When a sample exhibits more than one optical property, the elements of the Mueller matrix mix together and complicate interpretation. In general, the type of samples under examination can be organized by the sample's symmetry; isotropic, anisotropic, chiral and bi-anisotropic.

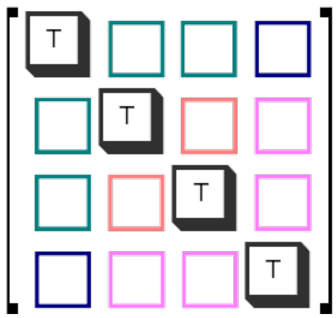


The Mueller matrix expressed with single optical effect labels.

T - tranmittance/reflectance
LE - linear extinction
CE - circular extinction
LR - linear retardance
CR - circular retardance
D - depolarization

Mueller matrix of isotropic materials

The most basic material is a simple isotropic material. In these materials, there is only absorbance. As such, only the diagonal of the Mueller matrix is occupied.

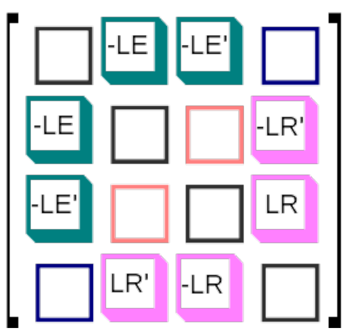


Isotropic samples exhibit a simple Mueller matrix with only diagonal elements. The sample's transmission occupies the diagonal and all values should be the same.

In general, most Mueller matrix polarimeters will normalize to M_{00}. As such, the Mueller matrix will measure as unity, one, along the diagonal. The isotropic Mueller matrix serves as a starting point to understand more complex samples; anisotropic and chiral samples.

Anisotropic Mueller matrices

Anisotropic materials poses different refractive indices in different directions. The refractive index is direction dependent and shows two or three distinct orthogonal refractive index values. Most anisotropic samples do not exhibit circular retardance, CR, and circular extinction, CE, due to space group restrictions. For these more complex samples, the values and signs of the diagonal elements depend heavily on which optical properties are expressed by a material. So for the sake of intuitive interpretation, it is suggested to largely ignore the diagonal values and focus on the signs and relationships of the off diagonal elements.



Most anisotropic samples in transmission will show only linear extinction and linear retardance.

Ignoring the diagonal elements allows for the Mueller matrix to be split into two segments; extinction/dichroism and retardance/birefringence. The first row and first column contain the majority of the information about differential extinction. For a simple homogeneous material, the signs and magnitude should match across the diagonal for M_{01} and M_{10} as well as M_{02} and M_{20}. Retardance falls into the 3x3 lower right cube. Here, the signs of M_{13} and M_{31} should be opposite as are the signs of M_{23} and M_{32}.

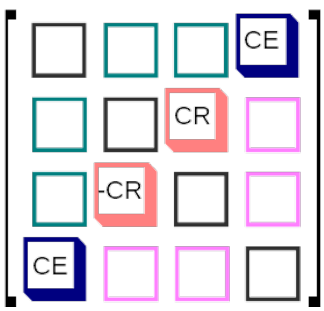
Anisotropic samples will poses large values for the Mueller matrix elements M_{12} and M_{21}. The values of these elements will have the same sign and magnitude as they are not due to circular retardance but the interactions of linear retardance and/or linear extinction. To distinguish between the effects of linear retardance and circular retardance on the Mueller matrix elements M_{12} and M_{21}, the circular retardance can be expressed in the following equation.

$$CR \sim \frac{M_{12} - M_{21}}{2}$$

Most anisotropic samples will exhibit large linear retardance which will swamp and obscure any circular retardance. In addition, due to space group restrictions most anisotropic materials can not possess circular retardance. Only isotropic chiral samples will exhibit measurable circular retardance or circular dichroism.

Isotropic chiral Mueller matrices

Isotropic chiral materials are a classification of materials with a refractive index that is the same in all directions, but the material interacts deferentially with left and right circularly polarized light. When the sample is chiral isotropic, the resulting Mueller matrix will exhibit non-zero anti-diagonal elements.

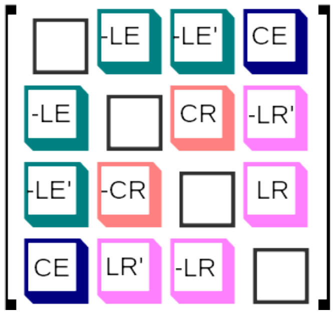


Chiral isotropic samples in transmission will normally only show the elements of circular extinction and circular retardance.

The elements M_{03} and M_{30} will have the same sign if the sample poses significant circular extinction. As noted above, circular retardance will be highlighted as the difference in the M_{12} and M_{21} elements. These two elements will show the opposite signs.

Bianisotropic Mueller matrices

Bianisotropic materials are the most complex classification of optical materials. These materials exhibit both linear and circular properties. For the most part, these materials will have very low symmetry such as the triclinic crystal system.



As most measured Mueller matrices are normalized, the absolute transmittance can not be measured. In addition the precise value of the diagonal elements depends heavily on the value of the optical properties.

For these materials, all the analysis above applies equally well. The orientation of the material's optical axes have a drastic effect on the measurable circular retardance. Due to the relative magnitudes of linear and circular retardance, the measurable circular retardance is generally swamped by linear retardance. The total retardance scales both the measurable linear and circular retardance by the sinc function.

$$TR = \sqrt{LB^2 + LB'^2 + CB^2}$$

$$CB_{measable} \sim CBsinc(TR)$$

$$LB_{measable} \sim LBsinc(TR)$$

Since linear retardance is usually orders of magnitude larger than circular retardance, circular retardance is generally only measurable in directions where the linear retardance is near zero. So measuring the circular retardance of an ansitropic sample is best done by measuring down the optical axes of an anisotropic material.

Analytical reduction of Mueller matrices to optical properties

Up to this point, the interaction between different optical properties was largely neglected in order to offer a simple analysis. With this simple analysis, the signs and relative relationship between Mueller matrix elements can be understood. For a precise analysis, the experimentally determined Mueller matrix can be assumed to arise from an arbitrarily complex homogeneous material. To analyze the optical properties, the Mueller matrix must be reduced to its basic six optical properties. This is most simply accomplished by applying the matrix logarithm.

$$m = -ln(M)$$

The numerical or analytical routines for a matrix logarithm are beyond the scope of this document. The technique separates the effects of each optical property, but does not take into account the phase order. As such, the matrix logarithm can reduce a Mueller matrix into an easily interpreted matrix containing the retardance and extinction information.

$$
m = \begin{bmatrix} \alpha & \beta & \gamma & \xi \\ \beta & \alpha & \mu & \nu \\ \gamma & -\mu & \alpha & \eta \\ \xi & -\nu & -\eta & \alpha \end{bmatrix}
$$

Where α is the isotropic absorption. β, γ, and ξ are the linear dichroism at 0°/90°, linear dichroism at 45°/135°, and circular dichroism respectively. μ, ν, and η are the linear birefringence at 0°/90°, linear birefringence at 45°/135°, and circular birefringence respectively.

[R. M. A. AZZAM ‘Propagation of partially polarized light through anisotropic media with or without depolarization: A differential 4x4 matrix calculus’ in: Journal of the Optical Society of America 68.12 (Dec. 1978), pp. 1756 –1767 DOI: 10.1364/JOSA.68.001756]

The differential Mueller matrix, *m*, can be experimentally determined by using numerical routines. The results can then be directly interpreted or converted to more familiar units. With these tools, a user can at a glance, interpret an experimentally determined Mueller matrix and gain insight into the optical properties of the material under examination.